Yeong-Chuan Kao and Yu-Wen Lee

Department of Physics, National Taiwan University, Taipei, Taiwan

In this paper, we study the Schwinger model on a half-line. In particular, we investigate the behavior the chiral condensate near the edge of the line. The effect of the chosen boundary condition is emphasized. The extension to the finite temperature case is straight forward in our approach.

I. INTRODUCTION

The (1+1)- dimensional massless spinor electrodynamics, the Schwinger model [1], has been a most popular playground for theorists because it exhibits many subtle properties which are believed to exist, yet difficult to verify, in four dimensional QCD. One such property is the existence of chiral condensate, an essential component in modern particle physics. In the Schwinger model, a rather complete understanding of the chiral condensate under various circumstances, e.g. at finite temperature and finite chemical potential [2–8], has been achieved. It is the unrealistic low dimensionality of the model that allows such detailed understanding. But sometimes realistic low-dimensional models can arise from (3+1)-dimensional ones if only radial dependence (s-wave approximation) is kept. For example, the famous Callan-Rubakov effect, i.e. the monopole catallysis of proton decay [10,9,11], made use of a Schwinger-like model defined on a half line in the first approximation. A fermion number breaking condensate analogous to the chiral condensate can be shown to exist around a magnetic monopole. Motivated partly by the Callan-Rubakov effect, we study the original Schwinger model on a half-line. We pay special attention to the dependence of the chiral condensate on the distance from the edge. Exact results can be obtained by using existing techniques. The standard value of the chiral condensate is recovered when we are in region far away from the edge. The value of the condensate near the edge depends on the chosen boundary condition. In this note, we adopt the boundary condition employed in the boundary conformal field theory approach to the Callan-Rubakov effect. It is easy to extend the result to the finite temperature case with our approach. Previous attempt to study the finite temperature behavior of the Callan-Rubakov condensate relied on the cluster decomposition property which is hard to implement on a half line [12].

II. BOSONIZATION OF QED$_2$ ON A HALF-LINE

We shall begin by first studying the Schwinger model on a finite segment. The Lagrangian density of QED$_2$ on a line segment of length L ($x \in [0, L]$) is defined by :

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}\gamma_\mu(i\partial_u - eA_\mu)\psi. \tag{2.1}$$

The boundary condition on ψ is chosen to be $\psi_R(t, L) + \psi_L(t, L) = 0 = \psi_R(t, 0) + \psi_L(t, 0)$ which breaks chiral symmetry at $x = 0$. [11] A similar boundary condition appears in the Callan-Rubakov effect and allows an exact treatment of the theory. Discussions on a generalization of this boundary condition will be given in the last section.

The half line case will be obtained if we take the infinite L limit at the end of the analysis. In two dimensions, it is possible to choose the Coulomb gauge so that A_x is independent of the spatial coordinate x, i.e. $\partial_x A_x = 0$. After making A_x independent of x, we can still make a topologically non-trivial gauge transformation $A'_x = A_x + iU^{-1}\partial_x U$, where $U(x) = e^{\frac{i2\pi nx}{L}}$. It is well known that these "large" gauge transformations are important in understanding the vacuum structure of QED$_2$. The true degrees of freedom of the gauge field are the zero modes. If we define $\frac{eL}{2\pi}A_x = W(t)$, the Wilson line operator $\exp(ie\int_0^L dxA_x) = \exp(ieLW(t))$ which transforms as $U^\dagger(L)\exp(ie\int_0^L dxA_x)U(0) = \exp(ie\int_0^L dxA_x)$ is gauge invariant and the gauge field and fermion fields transform under a $n = 1$ large gauge transformation as :

$$A_x \to A_x + \frac{2\pi}{eL}, \qquad \text{or} \qquad W(t) \to W(t) + 1,$$
$$\psi \to \exp(\frac{-2\pi ix}{L})\psi. \tag{2.2}$$

The time component of the gauge field A_0 is not a dynamical degree of freedom and can be determined to be $A_0(t, x) = -e\int_0^L dyD(x, y)j^0(t, y)$ by solving the Gauss's law :

$$E' = ej^0, \qquad \dot{E} = -ej^1. \tag{2.3}$$

$D(x, y)$ is the Green's function which satisfies $\partial_x^2 D(x, y) = \delta(x - y)$. Following standard procedures, the Hamiltonian reads :

$$\begin{aligned} H = \frac{F^2}{2L} &+ \int_0^L (\psi_R^\dagger i\partial_x\psi_R(x) - \psi_L^\dagger i\partial_x\psi_L(x)) \\ &+ \frac{2\pi}{L}W(t)\int_0^L dx(\psi_R^\dagger\psi_R(x) - \psi_L^\dagger\psi_L(x)) \\ &- \frac{e^2}{2}\int_0^L dx\int_0^L dyj^0(t, x)D(x, y)j^0(t, y), \end{aligned} \tag{2.4}$$

where $j^0(t, x) \equiv \psi_R^\dagger\psi_R(x) + \psi_L^\dagger\psi_L(x)$ The fermion field is written in terms of right- and left-moving components explicitly for latter convenience.

The Hamiltonian (2.4) is essentially a model of fermions interacting via long range Coulomb forces together with the quantum mechanical (topological nontrivial) degrees of freedom of the spatial component of the gauge field. We will empoly the methods developed for boundary critical phenomena [13,14]to study properties of the Schwinger model with open boundary conditions and calculate the chiral condensate analytically.

The first step of bosonizing the fermions with the chosen boundary conditions at the origin is mirror copying: our boundary condition $\psi_R(t,0) = -\psi_L(t,0)$ allows us to analytically continue the left-moving component of the fermion field to the domain $x < 0$ by defining $\psi_R(t,x) \equiv -\psi_L(t,-x)$ for $x \in [-L,0]$. Our theory can then be formulated as a theory of chiral (right-moving) fermion defined on the full segment ($x \in [-L,L]$) which leads to the full line in the infinite L limit. Our Hamiltonian then reads as :

$$\begin{aligned} H = & \frac{F^2}{2L} + \int_{-L}^{L} dx \psi_R^\dagger i\partial_x \psi_R(x) \\ & + \frac{2\pi}{L} W(t) \int_{-L}^{L} dx sgn(x) \rho_R(x) \\ & - \frac{e^2}{2} \int_0^L dx \int_0^L dy (\rho_R(x) + \rho_R(-x)) D(x,y) \\ & \times (\rho_R(y) + \rho_R(-y)). \end{aligned} \tag{2.5}$$

where $\rho_R(x) \equiv \psi_R^\dagger(x)\psi_R(x)$. We now bosonize the Hamiltonian by introducing a single chiral boson field $\hat{\phi}(x)$ with $\psi_R(x) = \frac{1}{\sqrt{2L}} : e^{i\hat{\phi}(x)} := \frac{1}{\sqrt{2\pi a}} e^{i\hat{\phi}(x)}$ where a is an ultraviolet cutoff [7]. Note that our boundary condition on $x = L$, $\psi_R(t,L) = -\psi_L(t,L)$, now implies that we have the periodic boundary conditions for the chiral fermion, i.e. $\psi_R(L) = \psi_R(-L)$. The chiral boson will therefore has the following mode expansion :

$$\begin{aligned} \hat{\phi}(t,x) &= q_R + \frac{2\pi p_R(t-x)}{2L} + \phi(t,x), \\ \phi(t,x) &= \sum_{q>0} \sqrt{\frac{\pi}{qL}} (e^{iqx - aq/2} b_q + h.c.), \end{aligned} \tag{2.6}$$

and $q = \frac{2\pi n}{2L}$ and the boson creation operators satisfying the usual commutation relations $[b_q, b_{q'}^\dagger] = \delta_{q,q'}$, $[q_R, p_R] = i$. The periodic boundary condition requires p_R to take integer values only. The meaning of various factors in the above formulas can be seen from the following bosonization identity:

$$\begin{aligned} \psi_R^\dagger(x)\psi_R(x) &= \frac{1}{2\pi}\partial_x\hat{\phi}(x) = \frac{p_R}{2L} + \frac{\partial_x\phi(x)}{2\pi} \\ \psi_L^\dagger(x)\psi_L(x) &= \frac{-1}{2\pi}\partial_x\hat{\phi}(-x) = \frac{p_R}{2L} - \frac{\partial_x\phi(-x)}{2\pi} \end{aligned} \tag{2.7}$$

so that $p_R = \int_0^L dx(\psi_R^\dagger\psi + \psi_L^\dagger\psi_L)$ is the charge operator while the chiral charge operator is $\int_0^L dx(\psi_R^\dagger\psi - \psi_L^\dagger\psi_L) = \frac{1}{\pi}(\phi(L) - \phi(0)) \equiv q_5$. Their transformation properties under the $n = 1$ large gauge transformation are

$$\begin{aligned} & W(t) \to W(t) + 1, \qquad p_R \to p_R. \\ & \phi(x) \to \phi(x) + \frac{2\pi}{L} \mid x \mid, \qquad q_5 \to q_5 - 2. \end{aligned} \tag{2.8}$$

Due caution must be taken to preserve gauge invariance in bosonizing the Hamiltonian which contains the kinetic energy term :

$$H_F = \frac{1}{4\pi} \int_{-L}^{L} dx (\partial_x\phi + \frac{2\pi p_R}{2L} + \frac{2\pi}{L} W sgn(x))^2. \tag{2.9}$$

While for the interaction term, we have:

$$\begin{aligned} H_{int} &= \frac{e^2}{8\pi^2} \int_0^L dx (\phi(x) - \phi(-x))^2 \\ &= \frac{e^2}{8\pi^2} \int_L^L dx (\phi(x)^2 - \phi(x)\phi(-x)). \end{aligned} \tag{2.10}$$

which is manifestly gauge invariant.

Together, we finally got the following bosonized Hamiltonian :

$$\begin{aligned} H = & \frac{F^2}{2L} + \frac{\pi}{2L} p_R^2 + \frac{2\pi}{L} W^2 + \frac{2W}{L} \cdot (\phi(L) - \phi(0)) \\ & + \frac{1}{4\pi} \int_{-L}^{L} dx \left[(\partial_x\phi(x))^2 + \frac{m^2}{2} (\phi(x)^2 - \phi(x)\phi(-x)) \right] \\ = & \frac{F^2}{2L} + \frac{2\pi}{L} p_R^2 + \frac{2\pi}{L} W^2 + \frac{2W}{L} \cdot \sum_{n=\text{odd}} \frac{-2}{\sqrt{n}} (b_n + b_n^\dagger) \\ & + \sum_{q>0} \frac{m^2/2 + q^2}{q} \left[b_q^\dagger b_q - \frac{m^2/2}{2(m^2/2+q^2)} (b_q b_q + b_q^\dagger b_q^\dagger) \right]. \end{aligned} \tag{2.11}$$

where $m^2 = e^2/\pi$ and $2W - q_5 \equiv Q_5$ is the gauge invariant chiral charge. Notice that the gauge fields couple only to the boundary degrees of freedom of the ϕ field. As the Hamiltonian contains only quadratic terms, we can solve for the ground state wave functional exactly. Although the topological nontrivial part of the gauge field couples to the boson field linearly, their contribution to the boson spectrum is of the order of $O(\frac{1}{L})$. We will therefore now concentrate on terms propotional to $b_q b_q$, $b_q^\dagger b_q^\dagger$ or $b_q^\dagger b_q$ in the Hamiltonian . This part of the Hamiltonian can be diagonalized easily via a Bogoliubov transformation

$$\begin{aligned} & b_q \equiv \cosh\varphi_q \ \hat{b}_q - \sinh\varphi_q \ \hat{b}_q^\dagger, \\ & \qquad \text{with} \qquad \tanh 2\varphi_q = \frac{-m^2/2}{m^2/2 + 2q^2}. \end{aligned} \tag{2.12}$$

The diagonalized Hamiltonian now reads:

$$H = \sum_{q>0} \sqrt{m^2 + q^2} \ \hat{b}_q^\dagger \hat{b}_q. \tag{2.13}$$

This shows clearly the well known fact that the spectrum contains a relativistic boson with mass m. The spectrum contain only positive momentum branch reflecting the fact that ϕ is a chiral boson. With this simple bosonic Hamiltonian and the bosonic representations of fermionic operators, we can easily calculate the fermion correlation functions.

III. CHIRAL CONDENSATE

The chiral condensate is given by $\bar{\psi}(x)\psi(x) = \psi_R^\dagger(x)\psi_L(x) + h.c.$. When written in terms of the chiral bosonic field it becomes :

$$\begin{aligned}
&\psi_R^\dagger(x)\psi_L(x) = -\psi_R^\dagger(x)\psi_R(-x)\\
&= \frac{-1}{2\pi a} e^{-i\hat{\phi}(x)} e^{i\hat{\phi}(-x)} = \frac{-1}{2\pi a} : e^{-i\phi(x)+i\phi(-x)} :\\
&\cdot e^{\langle 0|\phi(x)\phi(-x)-\frac{1}{2}(\phi(x)\phi(x)+\phi(-x)\phi(-x))|0\rangle}. \qquad (3.1)
\end{aligned}$$

To proceed , we need the following relations rom eq. (2.12):

$$\begin{aligned}
\langle 0 \mid b_q b_{q'} \mid 0\rangle &= -\cosh\varphi_q \sinh\varphi_{q'} \langle 0 \mid \hat{b}_q \hat{b}_{q'}^\dagger \mid 0\rangle\\
&\quad - \sinh\varphi_q \cosh\varphi_{q'} \langle 0 \mid \hat{b}_q^\dagger \hat{b}_{q'} \mid 0\rangle \ ,\\
\langle 0 \mid b_q b_{q'}^\dagger \mid 0\rangle &= \cosh\varphi_q \cosh\varphi_{q'} \langle 0 \mid \hat{b}_q \hat{b}_{q'}^\dagger \mid 0\rangle\\
&\quad + \sinh\varphi_q \sinh\varphi_{q'} \langle 0 \mid \hat{b}_q^\dagger \hat{b}_{q'} \mid 0\rangle \ ,\\
\langle 0 \mid b_q^\dagger b_{q'} \mid 0\rangle &= \sinh\varphi_q \sinh\varphi_{q'} \langle 0 \mid \hat{b}_q \hat{b}_{q'}^\dagger \mid 0\rangle\\
&\quad + \cosh\varphi_q \cosh\varphi_{q'} \langle 0 \mid \hat{b}_q^\dagger \hat{b}_{q'} \mid 0\rangle. \qquad (3.2)
\end{aligned}$$

With the above relations, we obtain

$$\begin{aligned}
&D_1(x) = \langle 0 \mid \phi(x)\phi(-x) \mid 0\rangle\\
&= \sum_{q>0} \frac{\pi}{qL} \left[-\sinh 2\varphi_q + \cosh 2\varphi_q \cdot \cos 2qx + i \sin 2qx\right] \ ,\\
&D_2(x) = \langle 0 \mid \phi(x)\phi(x) \mid 0\rangle\\
&= \sum_{q>0} \frac{\pi}{qL} \left[-\sinh 2\varphi_q \cdot \cos 2qx + \cosh 2\varphi_q\right] \ ,\\
&D_1(x) - D_2(x)\\
&= \sum_{q>0} \frac{\pi}{qL} \frac{q}{\sqrt{m^2+q^2}} (\cos 2qx - 1) + \sum_{q>0} \frac{\pi}{qL} \cdot i \sin 2qx.
\end{aligned} \qquad (3.3)$$

Using the following mathematical formulas :

$$\begin{aligned}
&\int_0^\infty dq \frac{\cos 2qx}{\sqrt{m^2+q^2}} = K_0(2mx)\\
&-\sum_{q=n\pi/L, n>0} \frac{e^{-aq}}{\sqrt{m^2+q^2}} = \gamma + \ln\frac{ma}{2} + O(\frac{1}{L})\\
&\sum_{q>0} \frac{\pi}{qL} e^{iqz-\alpha z} = \ln\frac{L}{\pi\alpha} + \ln\frac{\alpha}{\mid z \mid} + \frac{i\pi}{2} sgn(z),
\end{aligned} \qquad (3.4)$$

where K_0 is the modified Bessel function and γ is the Euler constant, we find

$$\frac{-1}{2\pi a} e^{D_1(x)-D_2(x)} = \frac{m}{4\pi} e^\gamma \cdot e^{K_0(2mx)}. \qquad (3.5)$$

Putting everything together, we finally get :

$$\langle \bar{\psi}(x)\psi(x)\rangle = \frac{m}{2\pi} e^\gamma \cdot e^{K_0(2mx)}. \qquad (3.6)$$

The above formula is valid for $L \to \infty$. When x is near the origin, the condensate becomes singular because of the chiral symmetry breaking boundary condition and the exact correlation of ψ_L and ψ_R at the origin. In regions far away from the origin, we expect that the condensate is not affected by the boundary condition. Indeed, when $x >> m$, $K_0(mx) \to 0$ and $\bar{\psi}(x)\psi(x) \to \frac{m}{2\pi}e^\gamma$ which is just the usual chiral condensate on an infinite line as it should be. [2–7]

The above result can be readily generalized to the finite temperature case. One only need to substitute $\langle \hat{b}_q^\dagger \hat{b}_q\rangle = 1 - \langle \hat{b}_q \hat{b}_q^\dagger\rangle = n(T) \equiv \frac{1}{e^{\omega(q)}-1}$ into eq. (3.2). Here $\omega(q) = \sqrt{m^2+q^2}$ is the energy dispersion for the boson excitations. Upon doing that, we have :

$$\begin{aligned}
D_1(q) - D_2(q) = \sum_{q>0} &[(1+2n(T))(\cosh 2\varphi_q + \sinh 2\varphi_q)\\
&\times (\cos 2qx - 1) + i \sin 2qx] \,,
\end{aligned} \qquad (3.7)$$

and the chiral condensate at finite temperature becomes :

$$\begin{aligned}
&\langle \bar{\psi}(x)\psi(x)\rangle_{T>0}\\
&= \langle \bar{\psi}(x)\psi(x)\rangle_{T=0} \cdot \exp \sum_{q>o} \frac{\pi}{L} \frac{2}{e^{\sqrt{m^2+q^2}/T}-1} \frac{\cos 2qx}{\sqrt{m^2+q^2}}\\
&\quad \exp \sum_{q>0} \frac{\pi}{L} \frac{-2}{e^{\sqrt{m^2+q^2}/T}-1} \frac{1}{\sqrt{m^2+q^2}}.
\end{aligned} \qquad (3.8)$$

For $mx >> 1$, the second factor approaches one and the above result is again coincide with the one obtained in the infinite-line case. Eq. (3.6) and eq. (3.8) are the main results of this paper. Inclusion of a finite chemical potential is also straight forward following a previous treatment for the infinite line case. [8]

IV. DISCUSSIONS

It is important to point out that because of the symmetry breaking boundary condition, a unique ground state has been selected among the usual degenerate vacua far away from the boundary. The analogous situation in the usual spin system with a boundary is familiar: If we impose the condition that spins on the boundary must all point to the same direction ,say up, the spin system will settle down to a state with spins all pointing up when temperature lower than the critical temperature is approaching zero; one unique ground state is selected. If we rotate the boundary spin, the orientation of the spins deep inside will also change accordingly. Likewise here, if we change the boundary condition to $\psi_R(t,0) = -e^{i\theta}\psi_L(t,0)$, the chiral condensate eq. (3.6) and eq. (3.8) will contain an extra factor $cos(\theta)$ where θ is the famous vacuum angle. To understand this, one only needs to look at eq. (3.1) and see that an extra factor, $e^{i\theta}$, must be put on the right hand side. The $cos(\theta)$ factor arises from combining $e^{i\theta}$ with $e^{-i\theta}$ that will thereby appears in the hermitian conjugate of eq. (3.1).

Summing up, we have demonstrated that the chiral condensate can be obtained exactly in the Schwinger model on a half line if we choose an appropriate (yet not unphysical) boundary condition. More general boundary conditions may not allow exact treatments.

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